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EXACT SOLUTIONS OF THE PROBLEM OF THE VIBRO-IMPACT OSCILLATIONS OF A DISCRETE SYSTEM WITH TWO DEGREES OF FREEDOM†

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Non-smooth time transformations are used to investigate strongly non-linear periodic free oscillations of a vibro-impact system with two degrees of freedom. Allowance for the boundary conditions at collision times enables the singularities induced by these transformations to be eliminated. The smoothed equations of motion turn out to be linear. Investigation of the periodic solutions reveals vibro-impact states with one- an two-sided collisions, including localized states (only one of the masses experiences collisions with stopping devices), and their bifurcation structure. © 1999 Elsevier Science Ltd. All rights reserved.

Investigations of vibro-impact oscillations and similar solutions have employed a variety of approaches, including Poincaré mappings [1, 2], reduction to boundary-value problems [3], numerical integration of the equations of motion [4], perturbation theory and other analytical methods [5–10]. In this paper, free periodic oscillations of a vibro-impact system with two degrees of freedom will be studied using non-smooth time transformations [7, 8], previously used to construct a modified perturbation theory for strongly non-linear systems [9–12]. The stability or instability of the selected periodic regimes was established by numerical integration of the equations of motion with initial data corresponding to the theoretical solutions. When this was done, in the stable case the numerical solution remained close to the theoretical one for a long time (compared with the periods of the natural oscillations of the linearized system), while in the unstable case rapid divergence was observed.

We will consider a system of two linear oscillators of unit mass and stiffness, linked together by a linearly elastic rod of stiffness ε. The amplitudes of the oscillations of both masses are limited by absolutely stiff stopping devices, symmetrically placed at unit distance from each mass. The equations of motion have the form

$$\ddot{u}_1 + u_1 + \varepsilon(u_1 - u_2) + P(u_1, \dot{u}_1) = 0, \quad \ddot{u}_2 + u_2 + \varepsilon(u_2 - u_1) + P(u_2, \dot{u}_2) = 0 \tag{1}$$
$$P(u_i, \dot{u}_i) = 2\dot{u}_i[\delta(u_i + 1) - \delta(u_i - 1)], \quad i = 1, 2$$

(the dot stands for differentiation with respect to time t).

In order to simplify the investigation of vibro-impact states, we introduce a non-smooth time transformation [7, 8]

$$\tau(t) = \frac{T}{\tau} arc \sin\left(\sin\frac{\pi t}{T}\right); \quad e(t) = \dot{\tau}(t) \tag{2}$$

where T is the half-period of the desired solution. With this definition, $\tau = \tau(t)$ is a saw-tooth function and its derivative (in the sense of the theory of generalized functions) is a rectangular cosine function. Expressing the derivatives of $u_i(t)$ in terms of the variable τ, we obtain

$$\dot{u}_i = u_i' e, \quad \ddot{u}_i = u_i'' e^2 - 2u_0' \sum_{k=-\infty}^{\infty} (-1)^{k+1}\delta(t - T_*), \quad T_* = \frac{T}{2} + kT, \quad e^2 = 1$$

(the prime stands for differentiation with respect to τ).

Consequently, the second derivative $\ddot{u}_i$ contains singular terms, which describe discontinuities at times $t = T_*$ and vanish everywhere except at these points. These discontinuities are unimportant if at these times $u_i' = 0$; otherwise, collisions of the ith mass with the stopping devices are associated with these times. But then the non-linear terms in the equations of motion may be written as follows:

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$$P(u_i, \dot{u}_1) = 2u_i' \sum_{k=-\infty}^{\infty} (-1)^{k+1} \delta(t - T_*)$$

Here, too, singularities occur at $t = T_*$, and again these are unimportant if $u_i' = 0$ at these times (note that the zeros of the function u_i' are also zeros of the function $\dot{u}_i$). It is obvious that these terms are compensated by the singular terms occurring in the expression for the second derivative $\ddot{u}_i$.

Thus, when investigating periodic states in which every collision of one of the masses with a stopping device corresponds to either a collision or a maximum displacement (without collision) of the second mass, the equations of motion, after the change of time, are linearized and take the form

$$u_1'' + u_1 + \varepsilon(u_1 - u_2) = 0, \quad u_2'' + u_2 + \varepsilon(u_2 - u_1) = 0 \tag{3}$$

Ignoring the trivial cases of in-phase and antiphase normal modes, when there are either no impacts or both masses collide simultaneously with the stopping devices, we consider system (3) with the following initial conditions

$$u_1(0) = 1, \quad u_2'(0) = 0 \tag{4}$$

(the origin of time τ is displaced to the point of the first collision, without modifying the notation). Then the corresponding solution may be written as follows:

$$u_{1,2}(\tau) = \frac{1+u_2(0)}{2}\cos\tau \pm \frac{1-u_2(0)}{2}\cos\alpha\tau + \frac{u_1^1(0)}{2}\sin\tau \pm \frac{u_1^1(0)}{2\alpha}\sin\alpha\tau \tag{5}$$

where $\alpha = \sqrt{(1+2\varepsilon)}$ is a coupling parameter, with the plus sign for u_1 and the minus sign for u_2.

Note that the solutions of Eqs (3), unlike those of the original system, exist only in the domain $|\tau| \leq 1$. As a matter of fact, we can define a vibro-impact solution over a half-period of the oscillations, solving the boundary-value problem, and then, taking the periodicity of the function $\tau = \tau(t)$ into account, periodically continue it to large time intervals. When this is done we have to consider three types of periodic vibro-impact oscillations, corresponding to the following boundary conditions

$$\begin{aligned} &\text{Type a: } u_1'(T) = 0, \quad u_2(T) = -1 \\ &\text{Type b: } u_1(T) = -1, \quad u_2'(T) = 0 \\ &\text{Type c: } u_1'(T) = 0, \quad u_2(T) = 1 \end{aligned} \tag{6}$$

In addition, allowance is made for condition (4) at $\tau = 0$. Substituting expressions (5) into (6), we obtain the following transcendental relations

$$\begin{aligned} &\frac{1+u_2(0)}{2}\cos T + I_u\frac{1-u_2(0)}{2}\cos\alpha T + \frac{u_1^1(0)}{2}\sin T + I_u\frac{u_1^1(0)}{2\alpha}\sin\alpha T = I_\varphi \\ &-\frac{1+u_2(0)}{2}\sin T - I_u\frac{1-u_2(0)}{2}\alpha\sin\alpha T + \frac{u_1^1(0)}{2}\cos T + I_u\frac{u_1^1(0)}{2\alpha}\cos\alpha T = 0 \end{aligned} \tag{7}$$

Depending on the type of vibro-impact oscillation, the coefficients take the following values

$$\begin{aligned} &\text{Type a: } I_u = -1, \; I_u = +1, \; I_\varphi = -1 \\ &\text{Type b: } I_u = +1, \; I_u = -1, \; I_\varphi = -1 \\ &\text{Type c: } I_u = -1, \; I_u = +1, \; I_\varphi = +1 \end{aligned} \tag{8}$$

Expressing $u_i'(0)$ in terms of $u_2(0)$ from the second equation of (7) and substituting the result into the first equation, we obtain an expression relating the half-period T and $u_2(0)$

$$u_2(0)(\cos T - I_u\cos\alpha T) + \cos T + I_u\cos\alpha T + u_1^1(0)\left(\sin T + \frac{I_u}{\alpha}\sin\alpha T\right) - 2I_\varphi = 0 \tag{9}$$

and moreover $\max_{0\leq\tau\leq 1}\{|u_1(\tau)|_1, |u_2(\tau)|\} \leq 1$.

Numerical solution of Eq. (9) with varied values of $u_2(0)$ reveals complicated periodic vibro-impact states of types a, b and c. In what follows we have analysed solutions with half-periods $0 \leqslant T \leqslant 2\pi$, but low-frequency states may be investigated in a similar way. The stability of the periodic orbits discovered here has been investigated numerically, by integrating the resulting equations of motion with theoretically predicted initial conditions.

The simplest case, from the standpoint of predicting periodic vibro-impact states, is that of a small coupling parameter ($\varepsilon \ll 1$). This has already been investigated in detail [3] and will therefore not be considered here.

Figure 1 presents the branches of the solution in the "total energy E—amplitude $u_2(0)$" plane for a value $\varepsilon = 1$ of the coupling parameter. Branches 1 and 2 represent stable periodic states (the solid curves) and unstable periodic states (the dashed curves) with one-sided impacts of one of the masses with its left stopping device, while the other mass collides with the right stopping device (type a), that is, the collisions take place with "unlike" stopping devices. The unstable branch 3 and stable branch 4 represent oscillations with two-sided collisions of the first mass (type b), the second mass experiencing no collisions. Finally, a solution 5 of type c (one-sided collisions of each of the masses with "like" stopping devices) is unstable. Some modes of oscillation are shown in the configuration plane (u_1, u_2) for $\varepsilon = 1$ in Fig. 2; stable modes are indicated by the solid curves and unstable modes by the dashed curves. Comparison of the curves in Figs 1 and 2 enables us to draw important conclusions as to the change in the order in which the different types of periodic states are "born" as the energy parameter of the system is increased.

Obviously, if the energies are sufficiently small, the stable modes will be the in-phase and antiphase normal modes without collisions, which are not being considered here. In the case of weak coupling ($\varepsilon = 0, 1$) with increased energy, "saddle point-node" bifurcation will give rise to two pairs of localized modes with two-sided collisions of one of the masses [3]. It turns out that in the case $\varepsilon = 1$, increasing the energy parameter gives rise, first of all (at a value of the parameter close to unity), to the formation of stable (1a) and unstable (2a) modes with one-sided impacts of the masses on "unlike" stopping devices (Fig. 1). Only a further increase in the energy parameter gives rise to two stable and two unstable modes of type b, at an energy parameter value close to 2 (only one pair is shown in Figs 1 and 2: one stable and one unstable mode). As to modes of type c with one-sided impacts on "like" stopping devices, these are unstable.

In order to demonstrate the increasingly complex behaviour of the system at large values of the coupling parameter, we present the data for periodic vibro-impact states at $\varepsilon = 10$ (Fig. 3—in the "total energy E—amplitude $u_2(0)$" plane; Fig. 4—in the configuration plane (u_1, u_2)). Here branches 1a and 2a (stable, shown in Fig. 3 by the solid curves), 3a–5a (unstable, shown by the dashed curves) represent oscillations of type a (collisions with "unlike" stopping devices). In branches 1a–3a, the half-period varies in the range $2 < T < 3$; in the other branches, the relevant range is $3/2 < T < 2$. Branches of type b (two-sided collisions of one of the masses) were observed. When $5/2 < T < 7/2$, two of these are unstable

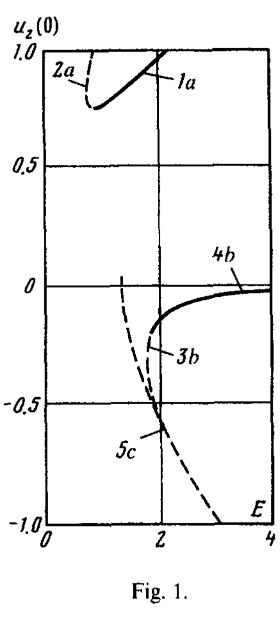

Fig. 1.

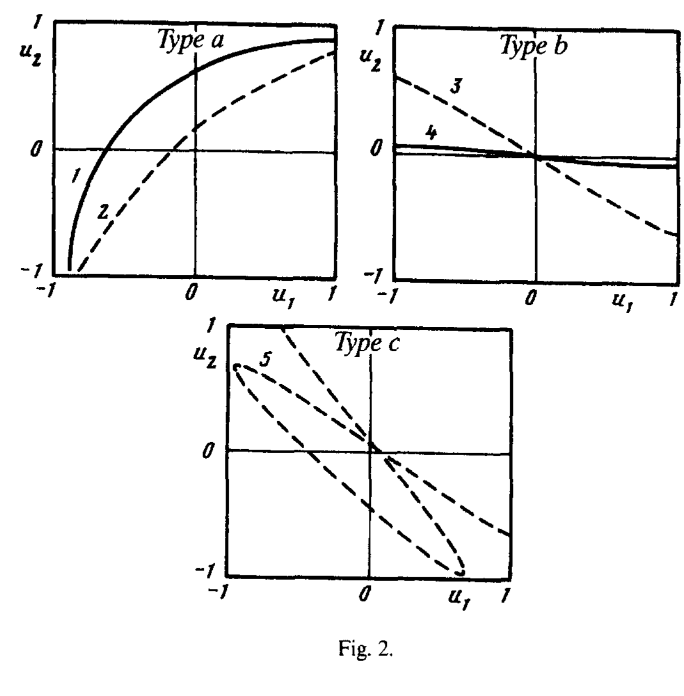

Fig. 2.

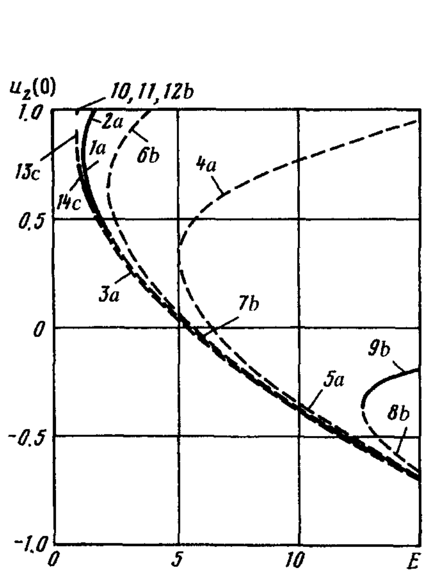

Fig. 3.

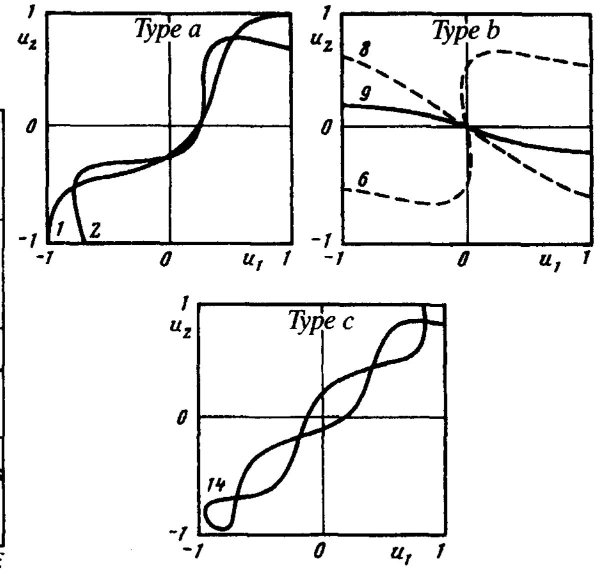

Fig. 4.

(10*b*, 12*b*) and one is stable (11*b*); when $1 < T < 5/s$, two are unstable (6*b*, 7*b*); finally, when $0 < T < 1$, one is unstable (8*b*) and one stable (9*b*). At the same time, unlike the case when $\varepsilon = 1$, a stable mode (14*c*) appears (one-sided collisions of the masses with "like" stopping devices) while the second mode of this type (13*c*) is unstable.

Figure 4 shows some of the stable and unstable modes of types *a*, *b* and *c* that have been found. The numbers on the curves in Fig. 4 correspond to the numbering of the modes in Fig. 3. Analysis of the results presented in Figs 3 and 4 indicates that the order in which modes of different types are "born" is again changed. In particular, stable localized modes, which correspond to two-sided collisions of one of the masses and are most important in the case of small ε values, may be realized here only at very large energies. However, modes with one-sided collisions of the masses with "like" stopping devices arise even at small energies.

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